Closing next wed: HW_2A,2B,2C

Office Hours: 1:30-3:00pm in Com.B-006

Quick review:

Def'n: The "signed" area between $f(x)$ and the x-axis from $x = a$ to $x = b$ is the *definite integral*:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

FTOC(1): Areas are antiderivatives!

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

FTOC(2): If F(x) is <u>any</u> antideriv. of $f(x)$,

$$\int_a^b f(x)dx = F(b) - F(a)$$

Entry Task: Evaluate

$$\int_0^4 e^x + \sqrt{x^3}\,dx \qquad x^{3/2}$$

$$= e^x + \frac{2}{5}x^{5/2}\Big|_0^4$$

$$= \left[e^4 + \frac{2}{5}(4)^{5/2}\right] - \left[e^0 + \frac{2}{5}(0)^{5/2}\right]$$

$$= e^4 + \frac{64}{5} - 1 = \boxed{e^4 + \frac{59}{5}}$$

$$\int_3^6 \frac{4}{x} - \frac{2}{x^2}\,dx = \int_3^6 4\frac{1}{x} - 2x^{-2}\,dx$$

$$= 4\ln|x| - 2\frac{1}{-1}x^{-1}\Big|_3^6$$

$$= 4\ln(x) + \frac{2}{x}\Big|_3^6$$

$$= \left[4\ln(6) + \frac{2}{6}\right] - \left[4\ln(3) + \frac{2}{3}\right]$$

$$= 4\ln(6) + 1/3 - 4\ln(3) - 2/3$$

$$= 4\ln(6) - 4\ln(3) - 1/3$$

$$= \boxed{4\ln(2) - 1/3}$$

5.4 The Indefinite Integral and Net/Total Change

Def'n: The **indefinite integral** of $f(x)$ is defined to be the general antiderivative of $f(x)$. And we write

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is any antiderivative of $f(x)$.

Ex)

$$\int x^3 dx = \frac{1}{4}x^4 + C \quad = \text{a function}$$

$$\int_0^1 x^3 dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4} \quad = \text{a number}$$

A brief pause to discuss current integration methods. We can currently find antiderivatives for sums and constant multiples of functions ***directly*** from our integration table.

Examples (we **can** currently do):

1. $\int 6e^x + 4x - 5\sqrt{x}\,dx$

$= 6(e^x) + 4(\frac{1}{2}x^2) - 5(\frac{2}{3}x^{3/2}) + C$

$= \boxed{6e^x + 2x^2 - \frac{10}{3}x^{3/2} + C}$

2. $\int 6\sec^2(x) - \frac{9}{x^4}\,dx$

$\int 6\sec^2(x) - 9x^{-4}\,dx$

$= 6(\tan(x)) - 9(\frac{1}{-3}x^{-3}) + C$

$= \boxed{6\tan(x) + \frac{3}{x^3} + C}$

4.9: LIST OF GENERAL ANTIDERIVATIVES

$\int 11^x dx$

$= \frac{1}{\ln(11)} 11^x + C$

FUNCTION	ANTIDERIVATIVE
$f(x) = x^n \ (n \neq -1)$	$F(x) = \dfrac{1}{n+1}x^{n+1} + C$
$f(x) = x^{-1} = \dfrac{1}{x}$	$F(x) = \ln\lvert x\rvert + C$
$f(x) = e^x$	$F(x) = e^x + C$
$f(x) = a^x$	$F(x) = \frac{1}{\ln(a)} a^x + C$
$f(x) = \cos(x)$	$F(x) = \sin(x) + C$
$f(x) = \sec^2(x)$	$F(x) = \tan(x) + C$
$f(x) = \underbrace{\sec(x)}_{}\tan(x)$, $\sec(x) = \frac{1}{\cos}$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$	$F(x) = \sec(x) + C$
$f(x) = \sin(x)$	$F(x) = -\cos(x) + C$
$f(x) = \csc^2(x)$	$F(x) = -\cot(x) + C$
$f(x) = \csc(x)\cot(x)$, $\csc(x) = \frac{1}{\sin(x)}$, $\cot(x) = \frac{\cos(x)}{\sin}$	$F(x) = -\csc(x) + C$
$f(x) = \dfrac{1}{1+x^2}$	$F(x) = \tan^{-1}(x) + C$

$e = 2.7182818$

$\frac{d}{dx}(2^x) = 2^x \ln/2$

$f(x) = a^x \qquad F(x) = \frac{1}{\ln(a)} a^x + C$

Examples we **cannot** currently do (but will be able to do later in the term):

$$\int xe^{3x}dx\,; \qquad \int \tan(x)dx$$

$$\int x\sin(x^2)\,dx; \qquad \int \frac{x^2-x+6}{x^3+3x}dx$$

$$\int \frac{3}{x-2\sqrt{x}}dx\,; \qquad \int \frac{\sqrt{x^2-1}}{x^2}dx$$

Examples we will *"never"* be able to do:

$$\int e^{x^2}dx\,; \int \sec(x^2)dx$$

Here are two that look bad but we can currently do them, why?

1. $\int \frac{\sqrt{x}-3x}{x}dx = \int \frac{1}{x}(\sqrt{x}-3x)dx$

2. $\int \frac{\cos(x)}{1-\cos^2(x)}dx$

$\boxed{1}$ $\int x^{-1/2}-3\,dx$

$= \boxed{2x^{1/2}-3x+C}$

$\boxed{2}$ $\int \frac{\cos(x)}{\sin^2(x)}dx$

$= \int \frac{\cos(x)}{\sin(x)}\frac{1}{\sin(x)}dx$

$= \int \cot(x)\csc(x)dx$

$= \boxed{-\csc(x)+C}$

What is the value of:

$$\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{1-\cos^2(\mathrm{x})}\, dx$$

$$= -\csc(x)\Big|_{\pi/4}^{\pi/2}$$

$$= \left[-\csc\left(\tfrac{\pi}{2}\right)\right] - \left[-\csc\left(\tfrac{\pi}{4}\right)\right]$$

$$= \left[-\frac{1}{\sin(\pi/2)}\right] - \left[-\frac{1}{\sin(\pi/4)}\right]$$

$$= -1 + \frac{1}{\sqrt{2}/2}$$

$$\boxed{\begin{aligned} &= -1 + \frac{2}{\sqrt{2}} \\ &= -1 + \sqrt{2} \end{aligned}}$$

Net Change and Total Change

Assume an object is moving along a straight line (up/down or left/right).

Let

$s(t)$ = ‘location at time t’

$v(t)$ = ‘velocity at time t’

pos. $v(t)$ means moving up/right

neg. $v(t)$ means moving down/left

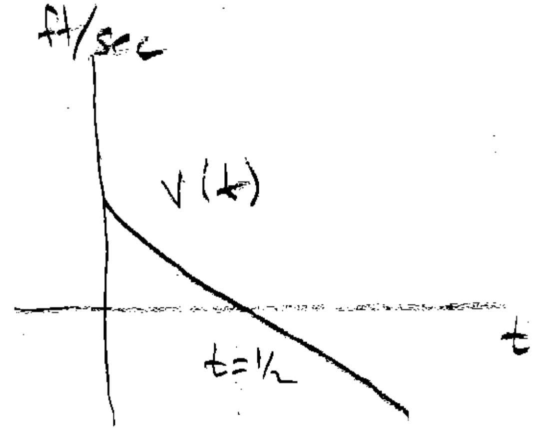
The FTOC (part 2) says

$$\int_a^b v(t)dt = s(b) - s(a)$$

i.e.

‘integral of velocity’= ‘**net change** in dist’

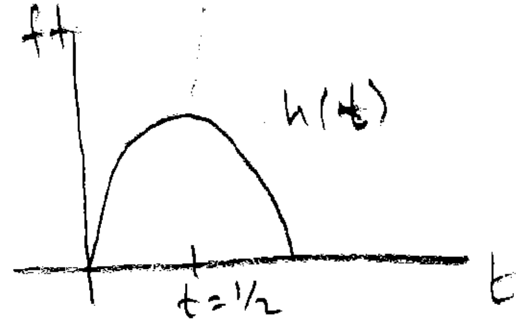
We also call this the *displacement*.



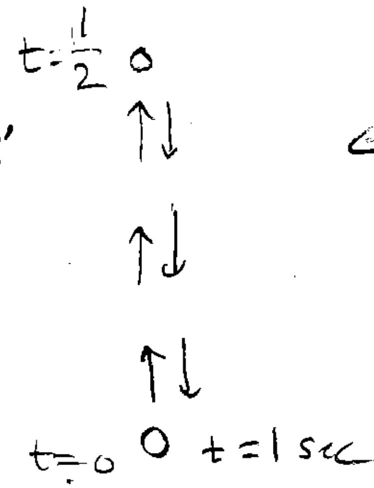
$a(t) = -32$

$v(t) = -32t + 16$

$h(t) = -16t^2 + 16t + 0$



ACTUAL VIEW OF OBJECT



$\int_0^1 v(t)dt = 0$

displacement

Thus, in general, the FTOC(2) says the **net change** in $f(x)$ from $x = a$ to $x = b$ is the integral of its **rate**. That is:

$$\int_a^b \underbrace{f'(t)}_{\frac{\text{gal}}{\text{hr}}}\,\underbrace{dt}_{\text{hr}} = \underbrace{f(b) - f(a)}_{\text{Change in gallons}}$$

$\frac{\text{people}}{\text{yr}}$ yr change in people

etc.

We define **total change** in dist. by

$$\int_a^b |v(t)|dt$$

which we compute by

1. Solving $v(t) = 0$ for t.
2. Splitting up the integral at these t values, dropping the absolute value and integrating separately.
3. Adding together as positive numbers.

Example: $v(t) = t^2 - 2t - 8$ ft/sec

Compute the total distance traveled from t = 1 to t = 6.

$$\int_1^6 |t^2-2t-8|\,dt$$

$\boxed{1}$ $t^2-2t-8 \stackrel{?}{=} 0$

$(t-4)(t+2)=0$

$t=4$ and $t=-2$

$\boxed{2}$ $\int_1^4 t^2-2t-8\,dt = -18$ ft

$\int_4^6 t^2-2t-8\,dt = \frac{44}{3} = 14.\overline{6}$ ft

The object goes "left" 18 feet then "right" $14.\overline{6}$ feet

For $\boxed{\text{TOTAL DISTANCE} = 18 + 14.\overline{6} = 32.\overline{6} \text{ ft}}$

NOTE: DISPLACEMENT $= \int_1^6 t^2-2t-8\,dt = -18+14.\overline{6}$

$= -3.\overline{3}$ ft

Close Wed: HW_2A, 2B, 2C (5.3,5.4,5.5)

5.5 The Substitution Rule

Entry Task (Motivation):

1. Find the following derivatives

Function	Derivative?
$\cos(x^2)$	$-\sin(x^2)\ 2x$
$\sin(x^4)$	$\cos(x^4)\ 4x^3$
$e^{\tan(x)}$	$e^{\tan(x)}\ \sec^2(x)$
$(\ln(x))^3$	$3(\ln(x))^2\ \frac{1}{x}$
$\ln(x^4+1)$	$\frac{1}{x^4+1}\ 4x^3$

2. Rewrite as integrals:

$$\int -\sin(x^2)\ \underline{2x\ dx} = \cos(x^2) + C$$

$$\int \cos(x^4)\ \underline{4x^3\ dx} = \sin(x^4) + C$$

$$\int e^{\tan(x)}\ \underline{\sec^2(x)\ dx} = e^{\tan(x)} + C$$

$$\int 3(\ln(x))^2\ \underline{\frac{1}{x}\ dx} = (\ln(x))^3 + C$$

$$\int \frac{1}{x^4+1}\ \underline{4x^3\ dx} = \ln(x^4+1) + C$$

3. Guess and check the answer to:

$$\int 7x^6 \sin(x^7)\, dx = \int \sin(u)\, du$$

$$= -\cos(u) + C$$

$$= \boxed{-\cos(x^7) + C}$$

$u = x^7$

$\frac{du}{dx} = 7x^6 \iff du = 7x^6 dx$

Observations:

1. We are reversing the “chain rule”.
2. In each case, we see
   “inside” = a function inside another
   “outside” = derivative of inside

To help us mechanically see these connections, we use what we call:

The Substitution Rule:
If we write u = g(x) and du = g’(x) dx, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Ex $\int e^{\tan(x)} \sec^2(x)dx$

$u = \tan(x)$
$du = \sec^2(x)dx$

$= \int e^u \, du$

$= e^u + C = \boxed{e^{\tan(x)} + C}$

Aside (you do not need to write this)

Some theory

Recall:

$$\int_a^b f(g(x))g'(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x$$

If we replace u = g(x), then we are "transforming" the problem from one involving *x* and *y* to one with *u* and *y*.

This changes ***everything*** in the set up. The lower bound, upper bound, widths, and integrand all change!

Recall from Math 124 that

$$g'(x) = \frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$$

(with more accuracy when Δx is small)

Thus, we can say that

$$g'(x)\Delta x \approx \Delta u$$

In other words, if the width of the rectangles using *x* and *y* is Δx, then the width of the rectangles using *u* and *y* is $g'(x)\Delta x$.

And if we write $u_i = g(x_i)$, then

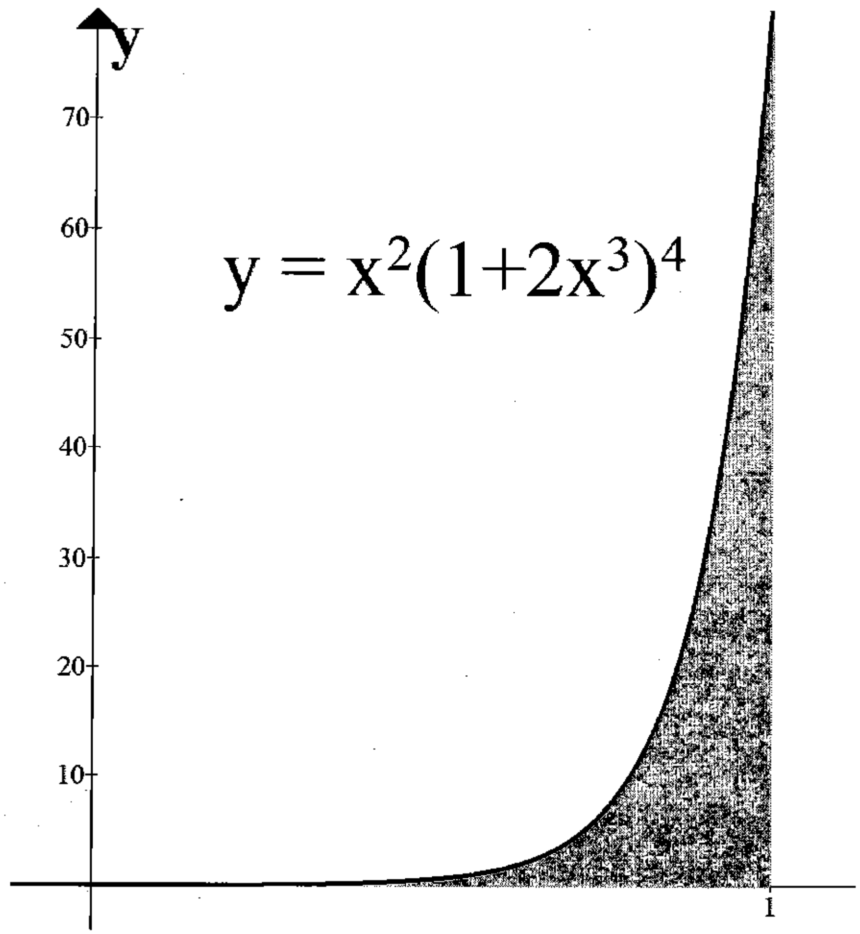
$$\int_a^b f(g(x))g'(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(u_i)\Delta u$$

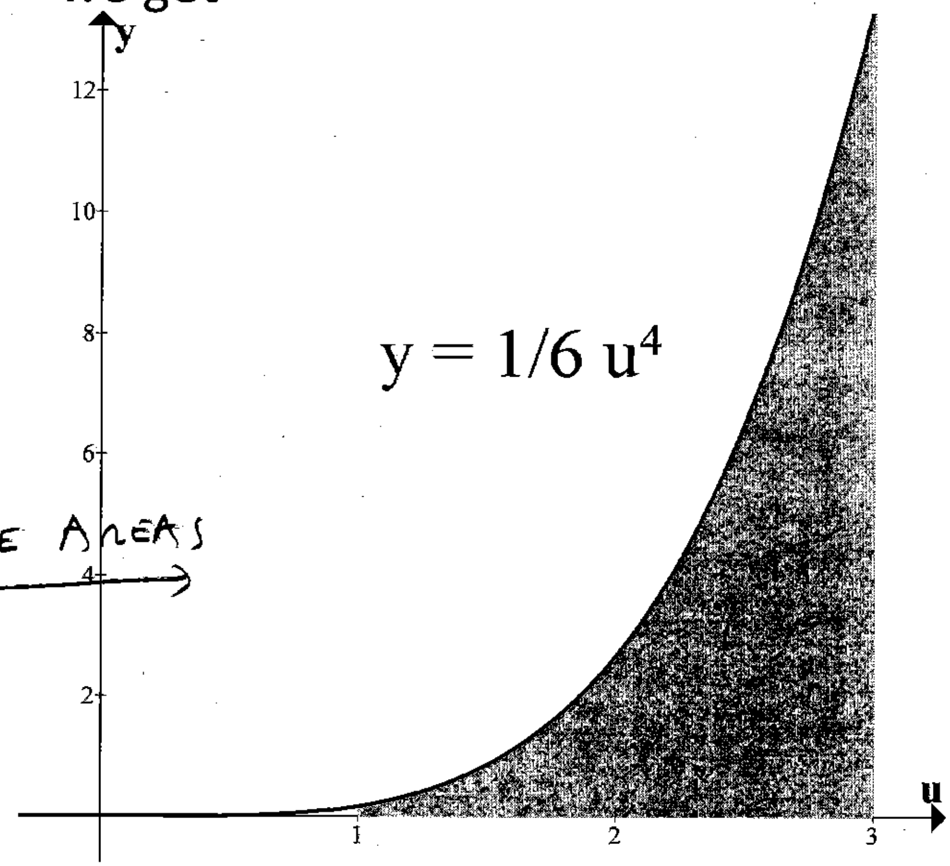
$$= \int_{g(a)}^{g(b)} f(u)du$$

Here is a visual example of this transformation

Using u = 1 + 2x³ and du = 6x² dx, we get



SAME AREAS



$$\int_0^1 x^2(1+2x^3)^4 dx$$

$u = 1 + 2x^3$

$du = 6x^2 dx$

$\frac{1}{6x^2} du = dx$

$$\int_1^3 \frac{1}{6} u^4 du$$

$\int_1^3 \cancel{x^2}\, u^4 \frac{1}{6\cancel{x^2}} du$

Examples:

First, try u = "inside function"

1. $\int x^4(1+x^5)^{31}\,dx$

$u = 1 + x^5$

$du = 5x^4\,dx$

$\frac{1}{5x^4}\,du = dx$

$= \int \cancel{x^4}\, u^{31} \frac{1}{5\cancel{x^4}}\,du$

$= \frac{1}{5}\int u^{31}\,du$

$= \frac{1}{5}\,\frac{1}{32}u^{32} + C$

$\boxed{\frac{1}{160}(1+x^5)^{32} + C}$

2. $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$= \int \frac{\sin(u)}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du$

$= 2\int \sin(u)\,du$

$= -2\cos(u) + C$

$= \boxed{-2\cos(\sqrt{x}) + C}$

3. $\displaystyle\int_2^3 x^2 e^{x^3}\,dx$

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3x^2}du = dx$

$= \displaystyle\int_8^{27} x^2 \; e^u \; \frac{1}{3x^2}\,du$

$= \displaystyle\frac{1}{3}\int_8^{27} e^u\,du$

$= \displaystyle\frac{1}{3}\left(e^u\Big|_8^{27}\right)$

$= \boxed{\frac{1}{3}\left(e^{27} - e^{8}\right)}$

4. $\displaystyle\int \frac{x\sin(x^2)}{\cos^2(\cos(x^2))}\,dx$

$u = \cos(x^2)$

$du = -2x\sin(x^2)\,dx$

$\frac{1}{-2x\sin(x^2)}du = dx$

$= \displaystyle\int \frac{\cancel{x\sin(x^2)}}{\cos^2(u)}\;\frac{1}{\cancel{-2x\sin(x^2)}}\,du$

$= \displaystyle -\frac{1}{2}\int \frac{1}{\cos^2(u)}\,du$

$= \displaystyle -\frac{1}{2}\int \sec^2(u)\,du$

$= -\frac{1}{2}\tan(u) + C$

$= \boxed{-\frac{1}{2}\tan(\cos(x^2)) + C}$

Examples:

Then, try u = "denominator"

1. $$\int_0^1 \frac{x}{x^2+3}\,dx$$

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$$= \int_3^4 \frac{\cancel{x}}{u}\frac{1}{2\cancel{x}}du$$

$$= \frac{1}{2}\int_3^4 \frac{1}{u}du$$

$$= \frac{1}{2}\ln|u|\Big|_3^4$$

$$\boxed{= \frac{1}{2}(\ln(4)-\ln(3)) = \frac{1}{2}\ln(4/3)}$$

2. $$\int \tan(x)\,dx$$

$$= \int \frac{\sin(x)}{\cos(x)}dx$$

$u = \cos(x)$

$du = -\sin(x)dx$

$\frac{1}{-\sin(x)}du = dx$

$$= \int \frac{\sin(x)}{u}\frac{1}{-\sin(x)}du$$

$$= -\int \frac{1}{u}du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

$$= \ln|(\cos(x))^{-1}| + C$$

$$= \ln\left|\frac{1}{\cos(x)}\right| + C$$

$$= \boxed{\ln|\sec(x)| + C = \int \tan(x)\,dx}$$

ADD TO TABLE OF KNOWN FACTS!

What to do when the "old" variable remains:
Examples:

1. $\int x^3\sqrt{2+x^2}\,dx$

$u = 2+x^2 \to x^2 = u-2$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$\int x^3 \sqrt{u} \frac{1}{2x} du$

$= \frac{1}{2}\int (u-2)\sqrt{u}\,du$

$= \frac{1}{2}\int u^{3/2} - 2u^{1/2}\,du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right)+c$

$= \boxed{\frac{1}{5}(2+x^2)^{5/2} - \frac{2}{3}(2+x^2)^{3/2}+c}$

2. $\int \frac{x^7}{x^4+1}dx$

$x^4 = u-1$

$u = x^4+1$

$du = 4x^3dx$

$\frac{1}{4x^3}du = dx$

$= \int \frac{x^{\cancel{7}\,4}}{u}\frac{1}{4\cancel{x^3}}du$

$= \frac{1}{4}\int \frac{u-1}{u}du$

$= \frac{1}{4}\int 1 - \frac{1}{u}du$

$= \frac{1}{4}(u - \ln|u|)+c$

$= \boxed{\frac{1}{4}(x^4+1) - \frac{1}{4}\ln|x^4+1|+c}$

Basic Integration Quiz Sheet

The following pages of integrals all can be evaluated by either simplification or u-substitution. The first 2 pages contain indefinite integrals. The last page contains definite integrals. By the end of the third week of class you should be able to complete the first 2 pages in 15-20 minutes and the last page in 15-20 minutes. So you should be able to complete these types of integral problems in about 1 minute or less each.

Note that our current methods are limited to these types of problems, there are lots of integrals we still are unable to do. (This means, on the first exam I can only ask you to evaluate integrals that can be completed using simplification or u-substitution.)

Evaluate all the following:

1. $\int 3x^{10} - \frac{\sqrt{x}}{x^2} + 4\,dx$

2. $\int dx$

3. $\int \sin(\tan(x))\sec^2(x)\,dx$

4. $\int x^7(1+x^8)^{31}\,dx$

5. $\int \tan(x) + \frac{\sin(x)}{\cos^2(x)} + 13xe^{x^2}\,dx$

6. $\int (5x^4 - 6x)\sqrt{x^5 - 3x^2 + 1}\,dx$

7. $\int x(1+x)^5 \, dx$

8. $\int \frac{\sqrt{3x^3} + 4x^2 - 11x}{\sqrt{x}} \, dx$

9. $\int \frac{x^5}{\sqrt{1+x^3}} \, dx$

10. $\int \cos(x)\sin(x) \, dx$

11. $\int \sin(13x) \, dx$

12. $\int e^{7x} \, dx$

13. $\int \cos(\frac{1}{4}x) \, dx$

14. $\int \sin(-5x) \, dx$

15. $\int e^{101x} \, dx$

16. $\int \cos(2x) \, dx$

17. $\displaystyle\int_0^{\left(\frac{\pi}{2}\right)^{1/3}} x^2 \sin(x^3)\ dx$

18. $\displaystyle\int_0^{\frac{\pi}{2}} e^{-3\cos(x)} \sin(x)\ dx$

19. $\displaystyle\int_{-10}^{-3} e^{\frac{1}{10}x}\ dx$

20. $\displaystyle\int_{\pi/3}^{\pi/4} \sin(-\frac{7}{8}x)\ dx$

21. $\displaystyle\int_2^3 \frac{x}{\sqrt{4+x^2}}\ dx$

22. $\displaystyle\int_1^5 \frac{x^3 - 2x^2 + x^{5/2}}{x^{1/3}}\ dx$

23. $\displaystyle\int_2^{2e} \frac{3-x}{2x}\ dx$

24. $\displaystyle\int_1^e \frac{(\ln(x))^3}{x}\ dx$